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On the asymptotic behavior of effective QED at finite temperature

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Abstract. The aim of this paper is to study finite temperature effects in effective quantum electrodynamics using Weisskopf's zero-point energy method in the context of thermo field dynamics. After a general calculation for a weak magnetic field at fixed T, the asymptotic behavior of the Euler–Kockel–Heisenberg Lagrangian density is investigated focusing on the regularization requirements in the high temperature limit. In scalar QED the same problem is also discussed.

1 Introduction

Finite temperature effects in effective quantum electrodynamics were first studied in the pioneering paper by Dittrich [1]. Motivated by the works of Weinberg [2] and others [3], who investigated finite temperature effects in a variety of models in quantum field theory, Dittrich performed a detailed analysis of the finite temperature effects on the spinor and scalar QED in the presence of a constant magnetic field. In particular, he examined the Euler–Heisenberg effective Lagrangian [4] at finite temperature in the context of the elegant Fock–Schwinger proper time method [5].

Almost ten years ago, Loewe and Rojas [6] extended Dittrich's work in the sense that they considered a general constant electromagnetic field. They found an effective thermal Euler–Heisenberg Lagrangian density from which the effective thermal coupling constant and an effective thermal mass for the photon were identified. While Dittrich made use of the so-called imaginary time formalism [7] in order to implement temperature effects, Loewe and Rojas employed the real time formalism known as thermo field dynamics (TFD) [8]. They argued that, despite the complication with doubling the number of fields, TFD allowed them to have a clear factorization of the interesting thermal effects. Along the same line, there are also the works by Elmfors et al. [9], in which the problem of charged fermions in a (weak) constant magnetic field was investigated. In agreement with [6], they found, for the high temperature limit, a logarithmic temperature dependence to the effective Lagrangian density.

In another important work on the issue of non-linear electromagnetic interactions at finite temperature, Brandt et al. [10] considered the analytically continued imaginary time thermal perturbation theory [11], showing on quite general grounds that, for high temperatures, the electron–positron box diagram contribution to the effective action has a finite non-zero limit. In addition, from arguments of Lorentz invariance, they also addressed the absence of the logarithmic temperature dependence in the non-relativistic static case as well as in the long wavelenght limit.

The purpose of this paper is to shed light on the problem of high temperature effective spinor and scalar QED, using Weisskopf's zero-point energy method [12] in the context of TFD, as well as to provide a unified connection between the present approach and the above referred ones, paying attention to the regularization requirements at the high temperatures.

In the next section we perform a general derivation of the weak magnetic field E-K-H Lagrangian density at a fixed temperature, analyzing a posteriori the high and low thermal limits. In Sect. 3, using the analytical regularization prescription [13] the high temperature contribution to the charge renormalization is addressed. In Sect. 4, using the Pauli–Villars–Rayski [14] prescription, the scalar QED is considered, and a closed expression for the high temperature E-K-H Lagrangian density is derived. Finally, Sect. 5 is devoted to some concluding remarks.

2 The E-K-H Lagrangian at finite temperature

The Dirac Hamiltonian for the electron–positron field in the presence of a constant and homogeneous electromagnetic field is written in Fock space as

$$\mathcal{H} = \sum_{\mathbf{p},\sigma} \epsilon^{(+)}_{\mathbf{p}\sigma} \left(a^{\dagger}_{\mathbf{p}\sigma} a_{\mathbf{p}\sigma} + b^{\dagger}_{\mathbf{p}\sigma} b_{\mathbf{p}\sigma} \right) + \varepsilon_0 \ , \qquad (1)$$

where

$$\varepsilon_0 = \langle 0|\mathcal{H}|0\rangle = -\sum_{\mathbf{p},\sigma} \epsilon_{\mathbf{p}\sigma} \qquad (2)$$

is the vacuum zero-point energy and $\epsilon_{\mathbf{p}\sigma}$ is the energy density related to the negative frequency solution of the Dirac equation in the presence of the electromagnetic field.

Following [15], the E-K-H effective Lagrangian density of the electromagnetic field, which accounts for the non-linear vacuum polarization effects, is given by

$$\delta\mathcal{L} = \mathcal{L}_{\text{ren}} - \mathcal{L}_0 = -\left[\varepsilon_0 - (\varepsilon_0)_{H=E=0}\right], \tag{3}$$

where $\mathcal{L}_0$ and $\mathcal{L}_{\text{ren}}$ are, respectively, the original and renormalized Lagrangian densities for the applied external electromagnetic field.

In order to study temperature effects in the E-K-H theory we make use of the so-called thermo field dynamics (TFD). In TFD, the temperature is introduced through a Bogoliubov transformation in the vacuum state of the system. In the present case, for each mode $(\mathbf{p}, \sigma)$, the vacuum state of the electron–positron field in the external electromagnetic field transforms as

$$\begin{aligned}|0\rangle &\to |0\rangle_\beta \\ &= (1+\mathrm{e}^{-\beta\epsilon})^{-1/2}\left\{|0\rangle_a \otimes |0\rangle_b + \mathrm{e}^{-\beta\epsilon/2} a^\dagger \tilde{a}|0\rangle_a \otimes b^\dagger \tilde{b}|0\rangle_b\right\},\end{aligned} \tag{4}$$

where $\tilde{a}$ and $\tilde{b}$ are annihilation operators of auxiliary fields which act on their corresponding vacuum sectors [8], and $\beta = (1/kT)$ (k is the Boltzmann constant and T the temperature).

Considering the case where only a magnetic field is present, the natural thermal generalization of (2) is

$$\begin{aligned}\langle 0|\mathcal{H}|0\rangle_\beta &= \varepsilon_0^\beta \\ &= \sum_{\mathbf{p}\sigma} \epsilon_{\mathbf{p}\sigma}\left\{\langle 0|a^\dagger a|0\rangle_\beta + \langle 0|b^\dagger b|0\rangle_\beta - \langle 0|1|0\rangle_\beta\right\} \\ &= \sum_{\mathbf{p}\sigma} \epsilon_{\mathbf{p}\sigma}\left\{\frac{1}{1+\mathrm{e}^{\beta\epsilon_{\mathbf{p}\sigma}}} - \frac{1}{1+\mathrm{e}^{-\beta\epsilon_{\mathbf{p}\sigma}}}\right\},\end{aligned} \tag{5}$$

where

$$\epsilon_{\mathbf{p},\sigma} = \sqrt{m^2 + (2n+1-\sigma)|e|H + p_z^2} \tag{6}$$

describes the energy levels of an electron with charge $-|e|$ in a constant and uniform magnetic field $H_z = -H$. In (6) $n = 0, 1, 2, ...$ and $\sigma = \pm 1$. Note that for $T \to 0$ the correct zero temperature result, expression (2), is recovered.

It must also be stressed that the above expression for the thermal zero-point energy was obtained from the TFD vacuum state (4), which, by construction, takes into account the interaction between the fermion field and a thermal reservoir.

Now, considering the density of momentum states $\frac{|e|H}{2\pi}\frac{\mathrm{d}p_z}{2\pi}$, the sum over the z-component of the electron momenta in (5) goes over to the continuum and the negative vacuum zero-point energy turns out to be

$$\varepsilon_0^\beta = \frac{|e|H}{(2\pi)^2}\sum_{n\sigma}\int_{-\infty}^{\infty}\mathrm{d}p\,\epsilon_{n\sigma}\left\{\frac{1}{1+\mathrm{e}^{\beta\epsilon_{n\sigma}}} - \frac{1}{1+\mathrm{e}^{-\beta\epsilon_{n\sigma}}}\right\}, \tag{7}$$

where we have already changed the notation $p_z \to p$. Expanding both geometric series in the curly brackets of (7) and taking the limit of weak magnetic field,

$$\begin{aligned}\varepsilon_0^\beta = \varepsilon_0 &+ \frac{2|e|H}{2\pi}\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\frac{\partial}{\partial\beta}\int_{-\infty}^{\infty}\mathrm{d}p\,\mathrm{e}^{-l\beta\sqrt{p^2+m^2}} \\ &\times\left\{-1+\left(\frac{2}{1-\mathrm{e}^{\frac{-l\beta|e|H}{\sqrt{p^2+m^2}}}}\right)\right\},\end{aligned} \tag{8}$$

where

$$\varepsilon_0 = -\frac{|e|H}{(2\pi)^2}\sum_{n\sigma}\int_{-\infty}^{\infty}\mathrm{d}p\,\epsilon_{n\sigma} \tag{9}$$

is the zero-point energy contribution which, ultimately, will give rise to the zero temperature E-K-H Lagrangian density. In the weak field limit,

$$\begin{aligned}\left(\frac{2}{1-\mathrm{e}^{\frac{-l\beta|e|H}{\sqrt{p^2+m^2}}}}\right) &\sim \frac{2\sqrt{p^2+m^2}}{l\beta|e|H} \\ &\times\left\{1-\frac{l\beta|e|H}{2\sqrt{p^2+m^2}}+\frac{5(l\beta|e|H)^2}{12(p^2+m^2)}\right\},\end{aligned} \tag{10}$$

so that (8) becomes

$$\begin{aligned}\varepsilon_0^\beta = \varepsilon_0 &+ \frac{8|e|Hm}{(2\pi)^2}\sum_{l=1}^{\infty}(-1)^{l+1}\frac{\partial^2}{\partial(l\beta m)^2}K_0(l\beta m) \\ &+ (\text{independent magnetic field term}) \\ &+ \frac{10|e|^2H^2\beta m}{3(2\pi)^2}\sum_{l=1}^{\infty}(-1)^{l+1}l\frac{\partial}{\partial(l\beta m)}K_0(l\beta m) \\ &+ \frac{10|e|^2H^2}{3(2\pi)^2}\sum_{l=1}^{\infty}(-1)^{l+1}K_0(l\beta m),\end{aligned} \tag{11}$$

where

$$K_0(l\beta m) = \int_0^{\infty}\mathrm{d}p\,\frac{\mathrm{e}^{-l\beta\sqrt{p^2+m^2}}}{\sqrt{p^2+m^2}} \tag{12}$$

is the modified Bessel function. From (11), it is straightforward to show that (3) gives

$$\begin{aligned}\delta\mathcal{L}^\beta = \delta\mathcal{L}_{\text{E-K-H}} &- \frac{2|e|Hm^2}{\pi^2}\sum_{l=1}^{\infty}(-1)^{l+1}K_0(l\beta m) \\ &- \frac{2|e|^2H^2m}{\pi^2\beta}\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}K_1(l\beta m) \\ &+ \frac{5|e|^2H^2}{6\pi^2}\sum_{l=1}^{\infty}(-1)^{l}K_0(l\beta m) \\ &+ \frac{5|e|^2H^2\beta m}{6\pi^2}\sum_{l=1}^{\infty}(-1)^{l+1}\,l\,K_1(l\beta m),\end{aligned} \tag{13}$$

where $K_1(x) = \frac{\partial K_0(x)}{\partial(x)}$. If, naively, the high temperature limit is taken in (13), a $\ln(\beta m)$ behavior may be found in virtue of the presence of the modified Bessel function. However, it must be noted that such a procedure is mathematically inconsistent since no regularization prescription was employed in (8). In fact, only in the low temperature regime, were the second term in (8) is regular, the above analysis makes any sense [16].

3 The high temperature E-K-H Lagrangian

In order to extract finite contributions from the divergent expression (7) (or (8)), when the high temperature limit is considered, the analytical regularization prescription [13] will be employed. Taking the $\beta \to 0$ limit in (7), we obtain

$$\varepsilon_0^{\beta\to 0} = -\frac{|e|H}{(2\pi)^2}\sum_{n\sigma}\int_{-\infty}^{\infty} \mathrm{d}p\ \epsilon_{n\sigma}\ \left\{\frac{6\beta(\alpha_{n\sigma}+p^2)^{1/2}}{12+\beta^2(\alpha_{n\sigma}+p^2)}\right\}, \tag{14}$$

where we have introduced $\alpha_{n,\sigma}$ through

$$\epsilon_{n\sigma}^2 = m^2 + (2n+1-\sigma)\,|e|H + p^2 = \alpha_{n\sigma} + p^2. \tag{15}$$

Now, to keep under control the ultraviolet divergence in (14), we make use of the gamma function integral representation, namely

$$\frac{1}{A^{1+\delta}} = \frac{1}{\Gamma(1+\delta)}\int_{0+}^{\infty} \mathrm{d}\eta\ \eta^{\delta}\mathrm{e}^{-A\eta}, \tag{16}$$

valid for $\delta > -1$, which allows us to make an analytical continuation by choosing an appropriated δ value. Hence, it follows that

$$\varepsilon_0^{\beta\to 0} = -\frac{|e|H}{(2\pi)^2}\frac{1}{\Gamma(1+\delta)}\sum_{n\sigma}\int_{0+}^{\infty}\mathrm{d}\eta\ \eta^{\delta}\mathrm{e}^{-\left(12+\beta^2\alpha_{n\sigma}\right)\eta} \times \left\{6\sqrt{\pi}\alpha_{n\sigma}\ \eta^{-1/2} - 3\eta^{-1}\frac{\partial}{\partial\beta}\left(\sqrt{\frac{\pi}{\beta^2\eta}}\right)\right\}, \tag{17}$$

where the gaussian integration over p has already been performed. Calculating the β derivative in the last term of (17) and making use of the identity

$$\sum_{n\sigma}\mathrm{e}^{-\beta^2\alpha_{n\sigma}\eta} = \mathrm{e}^{-\beta^2 m^2\eta}\left(1 + 2\sum_{n=1}^{\infty}\mathrm{e}^{-\beta^2(2|e|Hn)\eta}\right) = \mathrm{e}^{-\beta^2 m^2\eta}\coth(\beta^2|e|H\eta), \tag{18}$$

we rewrite (14) as

$$\varepsilon_0^{\beta\to 0} = A + B + C, \tag{19}$$

where

$$A = -\frac{|e|H}{(2\pi)^2}\frac{6\sqrt{\pi}m^2}{\Gamma(1+\delta)} \times \int_{0+}^{\infty}\mathrm{d}\eta\ \eta^{-3/2+1+\delta}\mathrm{e}^{-\left(12+\beta^2 m^2\right)\eta}\coth\left(\beta^2|e|H\eta\right), \tag{20}$$

$$B = -\frac{|e|H}{(2\pi)^2}\frac{6\sqrt{\pi}|e|H}{\Gamma(1+\delta)} \times \int_{0+}^{\infty}\mathrm{d}\eta\ \eta^{-3/2+1+\delta}\mathrm{e}^{-\left(12+\beta^2 m^2\right)\eta}\mathrm{csch}^2\left(\beta^2|e|H\eta\right) \tag{21}$$

and

$$C = \frac{|e|H}{(2\pi)^2}\frac{3\sqrt{\pi}}{\beta^2\Gamma(1+\delta)} \times \int_{0+}^{\infty}\mathrm{d}\eta\ \eta^{-3/2+\delta}\mathrm{e}^{-\left(12+\beta^2 m^2\right)\eta}\coth\left(\beta^2|e|H\eta\right). \tag{22}$$

The above integrals are finite objects and therefore the integrands in (20)–(22) can be expanded in a power series for $\beta m << 1$, within the convergence domain which corresponds to the analytic continuation $\delta > -1$. In order to recover the original theory we take the limit $\delta \to 0$, after performing the η-integration. As a result we obtain, up to $|e|^4$,

$$A \simeq \frac{3\sqrt{12}m^2}{\pi\beta^2}\left(1+\frac{1}{24}\beta^2m^2\right) - \frac{|e|^2H^2}{4(12)^{3/2}\pi}m^2\beta^2\left(1-\frac{3}{24}\beta^2m^2\right) + \frac{|e|^4H^4}{12(12)^{7/2}\pi}m^2\beta^6, \tag{23}$$

$$B \simeq -\frac{2(12)^{3/2}}{\pi}\frac{1}{\beta^4}\left(1+\frac{3}{24}\beta^2m^2\right) + \frac{|e|^2H^2}{2\sqrt{12}\pi}\left(1-\frac{1}{24}\beta^2m^2\right) - \frac{3|e|^4H^4\beta^4}{40(12)^{5/2}\pi}\left(1-\frac{5}{24}\beta^2m^2\right) \tag{24}$$

and

$$C \simeq -\frac{(12)^{3/2}}{\pi}\frac{1}{\beta^4}\left(1+\frac{3}{24}\beta^2m^2\right) + \frac{|e|^2H^2}{4\sqrt{12}\pi}\left(1-\frac{1}{24}\beta^2m^2\right) - \frac{|e|^4H^4\beta^4}{80(12)^{5/2}\pi}\left(1-\frac{5}{24}\beta^2m^2\right). \tag{25}$$

It must be noted that the above expressions still presents a singular behavior for $\beta \to 0$ as can be seen from the first terms in expressions (23)–(25). On the other hand, using (3), the E-K-H effective Lagrangian density becomes

$$\delta\mathcal{L}^{\beta\to 0} = \mathcal{L}_{\mathrm{ren}}^{\beta\to 0} - \mathcal{L}_0 = -\left[\varepsilon_0^{\beta\to 0} - (\varepsilon_0)_{H=0}^{\beta\to 0}\right] \simeq -\frac{|e|^2H^2}{8\pi}\left(\frac{6}{\sqrt{12}} - \frac{5}{(12)^{3/2}}m^2\beta^2\right) + \frac{|e|^4H^4}{\pi}\left(\frac{29m^2\beta^6}{96(12)^{7/2}} - \frac{7\beta^4}{80(12)^{5/2}}\right), \tag{26}$$

and, therefore,

$$\begin{aligned}\mathcal{L}_{\rm ren}^{\beta\to 0} &= \mathcal{L}_0 + \delta\mathcal{L}^{\beta\to 0} \\ &\simeq -\frac{|e|^2 H^2}{8\pi}\left[1+\left(\frac{6}{\sqrt{12}}-\frac{5}{(12)^{3/2}}m^2\beta^2\right)\right] \\ &= -\frac{H^2 e_{\rm ren}^2}{8\pi}, \end{aligned} \qquad (27)$$

where terms up to β^2 have been discarded and the renormalized charge was defined as

$$e \to e_{\rm ren}^{\beta\to 0} = e\left[1+\left(\frac{6}{\sqrt{12}}-\frac{5}{(12)^{3/2}}m^2\beta^2\right)\right]^{1/2}. \qquad (28)$$

4 Scalar QED

When spin effects become negligible, spinor QED reduces to scalar QED and a quite different theory then arises. In s-QED, the Hamiltonian for the charged boson field in the presence of an external electromagnetic field turns out to be

$$\mathcal{H} = \sum_{\mathbf{p}} \epsilon_{\mathbf{p}}^{(+)}\ \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}}\right) - \varepsilon_0\ , \qquad (29)$$

where

$$\varepsilon_0 = \langle 0|\mathcal{H}|0\rangle = -\sum_{\mathbf{p}} \epsilon_{\mathbf{p}}. \qquad (30)$$

Using the corresponding TFD vacuum state for each mode of the boson field

$$\begin{aligned}|0\rangle &\to |0\rangle_\beta \qquad (31)\\ &= \left(1+\mathrm{e}^{-\beta\epsilon}\right)^{1/2}\exp\left\{-\frac{\beta\epsilon}{2}\left(a^\dagger\tilde{a}^\dagger \otimes b^\dagger\tilde{b}^\dagger\right)\right\}|0\rangle_a \otimes |0\rangle_b\end{aligned}$$

and, again, considering the case where there is just a magnetic field, (30) becomes

$$\varepsilon_0^\beta = \langle 0|\mathcal{H}|0\rangle_\beta = \sum_{\mathbf{p}} \epsilon_{\mathbf{p}}\left\{\frac{-2+\mathrm{e}^{\beta\epsilon_{\mathbf{p}}}}{1-\mathrm{e}^{\beta\epsilon_{\mathbf{p}}}}\right\}, \qquad (32)$$

where

$$-\epsilon_{\mathbf{p}} = -\sqrt{m^2+2n|e|H+p^2} \qquad (33)$$

is the free spin degeneracy energy levels of the boson field with charge $-|e|$ in a constant and uniform magnetic field $H_z = -H$. Considering the corresponding density of momentum states, we have

$$\varepsilon_0^\beta = \frac{|e|H}{(2\pi)^2}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}p\ \epsilon_n\left\{\frac{-2+\mathrm{e}^{\beta\epsilon_n}}{1-\mathrm{e}^{\beta\epsilon_n}}\right\}. \qquad (34)$$

As in the fermion case, the above integral is difficult to solve analytically and we will restrict our analysis to its asymptotic behavior. At first, we take the low temperature limit. In this limit, it is easy to see that expression (34) leads to the expected zero temperature result. On the other hand, for $T \to \infty$ one can determine a priori, the high temperature effective Lagrangian density defined in (3). Taking $\beta \to 0$ in (34), we obtain

$$-\varepsilon_0^{\beta\to 0} = -\frac{|e|H}{(2\pi)^2}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}p\left\{\frac{1}{\beta}-\epsilon_n-\frac{\beta\epsilon_n^2}{2}\right\}, \qquad (35)$$

where terms up to second order in β were neglected.

The above momentum integral presents different degrees of divergences at the ultraviolet, which are more severe than in the spinor case, demanding not only a regularization prescription but also a consistent *subtraction* scheme. In order to extract finite results for the high temperature zero-point energy, we employ the Pauli–Villars–Rayski regularization prescription [14]. Following [16], where the motivation behind the use of such a regularization scheme is discussed in detail, we substitute (35) by its regularized expression

$$-\left(\varepsilon_0^{\rm R}\right)^{\beta\to 0} = -\sum_i c_i(\varepsilon_{0,i})^{\beta\to 0}, \qquad (36)$$

where

$$(\varepsilon_{0,i})^{\beta\to 0} = -\frac{|e|H}{(2\pi)^2}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}p\left\{\frac{1}{\beta}-\epsilon_{n,i}-\frac{\beta\epsilon_{n,i}^2}{2}\right\}. \qquad (37)$$

The linear, quadratic, cubic and logarithmic divergence appearing in (35) might be eliminated by imposing on the coefficients c_i the following conditions:

$$\sum_i^N c_i = 0, \qquad \sum_i^N c_i m_i^2 = 0. \qquad (38)$$

In (38), N is the total number of regulators and the coefficients are such that $c_0 = 1$ and $m_0 = m$ is the bare mass of the boson field.

From this scenario, the regularized high temperature effective Lagrangian density $\delta\mathcal{L}^{\rm R}$ is constructed following the same steps as done in the last section. However, despite the first term in (35) which does not contribute to $\delta\mathcal{L}^{\rm R}$ in virtue of the first condition in (38), the remaining two terms may be handled with the help of (16), keeping under control spurious *finite* β independent terms that violate the invariance of the theory under reflections $H \to -H$. This kind of odd-parity terms, which also appear in the zero temperature theory, are subtracted by the addition of finite counterterms with opposite sign. As a final result we obtain

$$\begin{aligned}\delta\mathcal{L} &= \frac{m^4}{8\pi^2}\int_{0^+}^{\infty}\mathrm{d}\eta\ \frac{\mathrm{e}^{-\eta}}{\eta^3}\left\{-\eta b\ \coth(\eta b)+1-\frac{1}{3}b^2\eta^2\right\} \\ &+ \frac{m^5\beta\sqrt{\pi}}{8\pi^2} \qquad (39)\\ &\times\int_{0^+}^{\infty}\mathrm{d}\eta\ \frac{\mathrm{e}^{-\eta}}{\eta^{7/2}}\left\{-\eta b\ \coth(\eta b)+1-\frac{1}{3}b^2\eta^2\right\},\end{aligned}$$

where $b = |e|H/m^2$, and a charge renormalization has been performed with the help of conditions (38).

5 Concluding remarks

In this paper we applied the formalism of thermo field dynamics in order to study temperature effects in the scope of spinor and scalar effective QED. Using Weisskopf's zero-point energy method, we constructed the high temperature E-K-H Lagrangian density, discussing the problem of charge renormalization at high temperatures.

In the spinor case, we first considered the weak magnetic field approximation in order to obtain, for a fixed temperature, a closed expression accounting for the thermal corrections to the E-K-H Lagrangian density. As can be seen from expression (13), the third and fourth terms correspond, respectively, to Dittrich's field (high) temperature contribution (the expression after (3.15) in [1]) and to Loewe and Rojas' thermal correction ((24) in [6]). In this sense, the connection between the effective action approach and Weisskopf's zero-point energy method would be helpful [17].

We have pointed out that (13) becomes mathematically inconsistent, in the high temperature limit, since no regularization was employed in (8). This issue was dealt with, in Sect. 3, using the analytic regularization scheme. As a result we have found that the renormalized coupling constant, expression (28), reaches a maximum slightly above its unrenormalized value as β approaches zero. This result is in complete agreement with that of Brandt et al. [10] in the sense that at high temperatures a finite non-zero contribution to the effective Lagrangian density is expected. We have also explicitly shown that for the case of a weak external magnetic field no logarithmic temperature dependence occurs. In fact, such a logarithmic behavior might be expected only in the low temperature limit [16], where it may be related to the ultraviolet divergence of the theory at zero temperature [10].

In Sect. 3, scalar QED was considered and a similar calculation has been performed. The divergences appearing in the asymptotic expression for the thermal zero-point energy (35) were regularized by means of the Pauli–Villars–Rayski *subtraction* scheme which, in contrast with the analytical method applied in Sect. 2, offers us a consistent way to deal with the divergent expression (35) as a whole object. In fact, since we have a priori no physical input at $\beta \to 0$ to eliminate spurious *infinite* contributions as that in the first term in (35) (which is field dependent and, therefore, is not subtracted when formula (3) is used) the P-V-R scheme had provided a natural way to overcome such a drawback through the specific conditions (38) imposed according to the different degrees of divergence appearing in each term of (35).

Finally, let it be emphasized that, despite the academic character of the subject, a better understanding of thermal effects in the context of effective QED is fundamental, since the range of applicability is broad [18]. Along these lines, we also conclude that Weisskopf's method in the context of TFD provides a powerful tool for the investigation of problems associated with radiative corrections in finite temperature quantum field theory.

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